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LETTER TO THE EDITOR

A rotor expansion of the su(3) Lie algebra†

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Abstract. Vector coherent state theory is used to give a rotor expansion of the su(3) Lie algebra in a way that parallels the boson expansions that have been made for other Lie algebras. The construction provides a systematic procedure for calculating Hermitian matrix elements of su(3) in an SO(3)-coupled basis and represents a new development in VCS theory and in the theory of induced representations.

It is well known that, for large-dimensional irreducible representations (irreps) the matrix elements of the su(3) Lie algebra approach those of corresponding representations of the rigid rotor algebra $[R^5]$so(3). By a new application of vector coherent state (VCS) theory (Rowe 1984, Hecht 1987), we show in this letter that any Hermitian su(3) irrep (corresponding to a unitary representation of the SU(3) group) can be realised in rotor terms. This is a useful result in view of the simplicity of the rotor algebra and the facility that has been developed in its use through applications to nuclear rotational states. The rotor expansion that we obtain for the su(3) Lie algebra provides a systematic procedure for the calculation of su(3) matrices in an so(3)-coupled basis. It provides a direct relationship between the su(3) and rigid-rotor descriptions of rotational states of deformed nuclei and it serves as a prototype for parallel rotor-type expansions of other Lie algebras. Thus, it represents a significant advance in VCS theory, which has already proved to be a powerful and versatile tool in the construction of induced representations.

The $[R^5]$so(3) Lie algebra is spanned by five commuting quadrupole moments $\{Q_\nu;\ \nu=0,\pm1,\pm2\}$ and three components $\{L_k;\ k=0,\pm1\}$ of angular momentum. A generic irrep of the rotor algebra (Ui 1970, Weaver *et al* 1973) is labelled by $\{\varepsilon, \varepsilon', q_0, q_2\}$ where ε and ε' take values ±1 and q_0, q_2 are positive real numbers which have the physical significance of intrinsic quadrupole moments. An orthonormal basis for such an irrep is given by the combinations

$$\psi_{KLM}(\Omega)=\left(\frac{2L+1}{16\pi^2(1+\delta_{K0})}\right)^{1/2}\{\mathscr{D}^L_{KM}(\Omega)+\varepsilon(-1)^{L+K}\mathscr{D}^L_{-KM}(\Omega)\} \tag{1}$$

of SO(3) Wigner functions with K restricted to either even or odd non-negative integer values such that $(-1)^K=\varepsilon'$. The rotor operators act on these wavefunctions by

$$\begin{aligned} &L_0\psi_{KLM}(\Omega)=M\psi_{KLM}(\Omega)\\ &L_\pm\psi_{KLM}(\Omega)=\sqrt{(L\mp M)(L\pm M+1)}\,\psi_{KLM\pm1}(\Omega)\\ &Q_\nu\psi_{KLM}(\Omega)=[q_0\mathscr{D}^2_{0\nu}(\Omega)+q_2(\mathscr{D}^2_{2\nu}(\Omega)+\mathscr{D}^2_{-2\nu}(\Omega)]\psi_{KLM}(\Omega). \end{aligned} \tag{2}$$

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In addition to the generic (so-called triaxial) representations, there are axially symmetric representations for which $q_2=0$. The latter are characterised by constant values of $\{\varepsilon, K, q_0\}$.

A basis for the complex extension of the u(3) Lie algebra is given by the nine operators $\{C_{ij};\ i, j=1,2,3\}$ which satisfy the commutation relations

$$[C_{ij}, C_{kl}]=\delta_{jk}C_{il}-\delta_{il}C_{kj}.$$

In terms of these operators, the su(3) subalgebra is spanned by five quadrupole moments

$$\begin{aligned}\mathcal{Q}_0&=2C_{11}-C_{22}-C_{33}\\ \mathcal{Q}_{\pm1}&=\mp\sqrt{\tfrac{3}{2}}(C_{12}+C_{21}\pm \mathrm{i}C_{13}\pm \mathrm{i}C_{31})\\ \mathcal{Q}_{\pm2}&=\sqrt{\tfrac{3}{2}}(C_{22}-C_{33}\pm \mathrm{i}C_{23}\pm \mathrm{i}C_{32})\end{aligned}\tag{3}$$

and three components of angular momentum

$$L_0=-\mathrm{i}(C_{23}-C_{32})\qquad L_{\pm1}=-\mathrm{i}(C_{31}-C_{13})\pm(C_{12}-C_{21}).\tag{4}$$

But, unlike the rotor quadrupole moments, the su(3) quadrupole moments do not commute. This reflects the fact that su(3) is compact whereas $[R^5]$so(3) is non-compact. Thus su(3) has finite-dimensional Hermitian irreps whereas those of $[R^5]$so(3) are infinite-dimensional. Nevertheless it is observed (Elliott 1958) that, for su(3) irreps with large highest weights (λ, μ), the states of angular momentum $L\ll\lambda$, $L\ll\mu$ are in one-to-one correspondence with those of rigid-rotor irreps. Furthermore, as $\lambda, \mu\to\infty$, the matrix elements of the su(3) quadrupole operators between such states approach those of a rigid rotor with

$$q_0=2\lambda+\mu\qquad q_2=\sqrt{\tfrac{3}{2}}\mu\qquad \varepsilon=(-1)^\lambda\qquad \varepsilon'=(-1)^\mu.$$

We show here that all states of any finite-dimensional su(3) irrep can be realised within a subspace of a rigid-rotor representation space and that on this irreducible su(3) subspace the su(3) quadrupole operators are represented by the rotor-like operators

$$\Gamma(\mathcal{Q}_\nu)=(2\lambda+\mu+3)\mathcal{D}^2_{0\nu}(\Omega)-\tfrac{1}{2}[L^2, \mathcal{D}^2_{0\nu}(\Omega)]+\sqrt{\tfrac{3}{2}}[\mathcal{D}^2_{2\nu}(\Omega)(\mu-\bar{L}_0)+\mathcal{D}^2_{-2\nu}(\Omega)(\mu+\bar{L}_0)]\tag{5}$$

where L^2 is the square of the angular momentum and $\bar{L}_0$ is a so-called intrinsic (i.e. left) angular momentum operator; cf (8).

Let V be the carrier space for a Hermitian su(3) irrep T having highest weight (λ, μ) and let $|\varphi\rangle\in V$ be the highest weight state. Then $|\varphi\rangle$ satisfies the equations

$$\begin{aligned}&T(C_{ij})|\varphi\rangle=0\qquad i<j\\ &T(C_{11}-C_{22})|\varphi\rangle=\lambda|\varphi\rangle\\ &T(C_{22}-C_{33})|\varphi\rangle=\mu|\varphi\rangle.\end{aligned}\tag{6}$$

Let $R(\Omega)$ denote the rotation operator for $\Omega\in \mathrm{SO}(3)$ generated by the representation T of the angular momentum operators of so(3).

We now recall the well known result (used by Elliott 1958) that the set of states

$$\{R(\Omega)|\varphi\rangle;\ \Omega\in\mathrm{SO}(3)\}$$

obtained by rotation of the highest-weight state $|\varphi\rangle$ through all possible angles, spans the su(3) representation space V. It follows that a state $|\psi\rangle\in V$ is uniquely defined by the 'coherent state' wavefunction

$$\psi(\Omega)=\langle\varphi|R(\Omega)|\psi\rangle\qquad \Omega\in\mathrm{SO}(3).$$

The wavefunction $\psi(\Omega)$ clearly belongs to the space $\mathscr{L}^2(\mathrm{SO}(3))$ of functions on SO(3) that are square integrable with respect to the SO(3)-invariant measure. Thus we obtain an embedding of the carrier space V for the hermitian su(3) irrep T in $\mathscr{L}^2(\mathrm{SO}(3))$ by

$$\mathscr{V} \to \mathscr{L}^2(\mathrm{SO}(3)) \qquad |\psi\rangle \mapsto \psi(\Omega).$$

Under this map the su(3) irrep T maps to an equivalent representation Γ defined by

$$T(X)|\psi\rangle \to \Gamma(X)\psi(\Omega) = \langle\varphi|R(\Omega)T(X)|\psi\rangle \qquad \forall X \in \mathrm{su}(3).$$

The representations T and Γ are clearly isomorphic. Therefore Γ, like T, is irreducible. It is a new kind of coherent state representation.

Under a rotation $\omega \in \mathrm{SO}(3)$, a state $|\alpha LM\rangle \in V$ of angular momentum L and z-component M transforms

$$|\alpha LM\rangle \to R(\omega)|\alpha LM\rangle = \sum_N |\alpha LN\rangle \mathscr{D}^L_{NM}(\omega).$$

The state $|\alpha LM\rangle$ is therefore represented by the wavefunction

$$\psi_{\alpha LM}(\Omega) = \langle\varphi|R(\Omega)|\alpha LM\rangle = \sum_N \langle\varphi|\alpha LN\rangle \mathscr{D}^L_{NM}(\Omega)$$

and transforms under $\omega \in \mathrm{SO}(3)$:

$$\psi_{\alpha LM}(\Omega) \to \langle\varphi|R(\Omega)R(\omega)|\alpha LM\rangle = \sum_N \psi_{\alpha LN}(\Omega)\mathscr{D}^L_{NM}(\omega).$$

Thus the representation Γ of the angular momentum operators acts in the standard way by

$$\begin{aligned}\Gamma(L_0)\psi_{\alpha LM}(\Omega) &= M\psi_{\alpha LM}(\Omega) \\ \Gamma(L_\pm)\psi_{\alpha LM}(\Omega) &= \sqrt{(L\mp M)(L\pm M+1)}\,\psi_{\alpha LM\pm1}(\Omega).\end{aligned} \tag{7}$$

Since $\mathscr{Q}_\nu$ is the component of a second-rank SO(3) tensor, it transforms under rotations

$$R(\Omega)\mathscr{Q}_\nu R(\Omega^{-1}) = \sum_\mu \mathscr{Q}_\mu \mathscr{D}^2_{\mu\nu}(\Omega).$$

It follows that

$$T(\mathscr{Q}_\nu)|\psi\rangle \to \Gamma(\mathscr{Q}_\nu)\psi(\Omega) = \sum_\mu \langle\varphi|\mathscr{Q}_\mu R(\Omega)|\psi\rangle \mathscr{D}^2_{\mu\nu}(\Omega).$$

From the highest-weight properties (6) of $|\varphi\rangle$, and the identities (3) and (4), it follows that

$$\begin{aligned}\langle\varphi|\mathscr{Q}_0 R(\Omega)|\psi\rangle &= (2\lambda+\mu)\langle\varphi|R(\Omega)|\psi\rangle = (2\lambda+\mu)\psi(\Omega) \\ \langle\varphi|\mathscr{Q}_{\pm1} R(\Omega)|\psi\rangle &= -\sqrt{\tfrac{3}{2}}\langle\varphi|L_\pm R(\Omega)|\psi\rangle = -\sqrt{\tfrac{3}{2}}\bar{L}_\pm\psi(\Omega) \\ \langle\varphi|\mathscr{Q}_{\pm2} R(\Omega)|\psi\rangle &= \sqrt{\tfrac{3}{2}}\langle\varphi|(\mu\mp L_0)R(\Omega)|\psi\rangle = \sqrt{\tfrac{3}{2}}(\mu\mp\bar{L}_0)\psi(\Omega)\end{aligned}$$

where $\bar{L}_0$ and $\bar{L}_\pm$ are the infinitesimal generators of left rotations. Their actions are defined by considering, for example,

$$\langle\varphi|L_k R(\Omega)|\alpha LM\rangle = \sum_N \langle\varphi|L_k|\alpha LN\rangle \mathscr{D}^L_{NM}(\Omega)$$

from which we infer that

$$\begin{aligned}\bar{L}_\pm \mathscr{D}^L_{NM}(\Omega) &= \sqrt{(L\pm N)(L\mp N+1)}\,\mathscr{D}^L_{N\mp1,M}(\Omega) \\ \bar{L}_0 \mathscr{D}^L_{NM}(\Omega) &= N\mathscr{D}^L_{NM}(\Omega)\end{aligned} \tag{8}$$

We conclude that

$$\Gamma(\mathcal{Q}_\nu) = (2\lambda+\mu)\mathcal{D}^2_{0\nu}(\Omega) - \sqrt{\tfrac{3}{2}}[\mathcal{D}^2_{1\nu}(\Omega)\bar{L}_+ + \mathcal{D}^2_{-1\nu}(\Omega)\bar{L}_-] + \sqrt{\tfrac{3}{2}}[\mathcal{D}^2_{2\nu}(\Omega)(\mu-\bar{L}_0) + \mathcal{D}^2_{-2\nu}(\Omega)(\mu+\bar{L}_0)]. \tag{9}$$

Finally, from the identity

$$[L^2, \mathcal{D}^2_{0\nu}(\Omega)] = 6\mathcal{D}^2_{0\nu}(\Omega) + 2\sqrt{\tfrac{3}{2}}[\mathcal{D}^2_{1\nu}(\Omega)\bar{L}_+ + \mathcal{D}^2_{-1\nu}(\Omega)\bar{L}_-]$$

we obtain (5).

It remains to determine the domain of the coherent state representation Γ; i.e. the subspace of coherent states in $\mathcal{L}^2(\mathrm{SO}(3))$ of the form $\psi(\Omega) = \langle\varphi|R(\Omega)|\psi\rangle$ with $|\psi\rangle \in V$. Let $\bar{\Gamma}(X)$, with $X \in \mathrm{su}(3)$, denote the extension of the operator $\Gamma(X)$, given by (5), to the dense subspace of $\mathcal{L}^2(\mathrm{SO}(3))$ spanned by all SO(3) irreps. Unlike Γ, which by definition is isomorphic to the irrep T and hence irreducible, the extended representation $\bar{\Gamma}$ is infinite dimensional and reducible. However, it is easily shown that the desired coherent state subspace is an invariant subspace of $\bar{\Gamma}$ and that Γ is the restriction of $\bar{\Gamma}$ to this subspace. Therefore, since every su(3) irrep has a single and hence uniquely identifiable multiplet of minimum angular momentum states $|L_{\min}M\rangle$, $M=-L,\ldots,+L$ with $L_{\min}=0$ or 1, one can generate the coherent state subspace by the repeated action of the $\bar{\Gamma}$ operators on these states.

First observe that a state $|0\rangle$ of $L=0$ has coherent state wavefunction

$$\psi(\Omega) = \langle\varphi|R(\Omega)|0\rangle = \langle\varphi|0\rangle$$

independent of Ω and hence proportional to the unique $K=L=0$ state $\psi_{00}(\Omega)$ of (1). A state $|1M\rangle \in V$ of angular momentum $L=1$ has wavefunction

$$\psi_{1M}(\Omega) = \langle\varphi|R(\Omega)|1M\rangle = \sum_N \langle\varphi|1N\rangle\mathcal{D}^1_{NM}(\Omega).$$

Now it is easily shown that under rotations through angle π about the y and z axes

$$R_z(\pi)|\varphi\rangle = (-1)^\mu|\varphi\rangle \qquad R_z(\pi)|1N\rangle = (-1)^N|1N\rangle$$
$$R_y(\pi)|\varphi\rangle = (-1)^\lambda|\varphi\rangle \qquad R_y(\pi)|1N\rangle = (-1)^{N-1}|1,-N\rangle.$$

Therefore

$$\begin{aligned}\psi_{1M}(\Omega) &= \langle\varphi|10\rangle\mathcal{D}^1_{0M}(\Omega) && \mu \text{ even}\\ &= \langle\varphi|11\rangle[\mathcal{D}^1_{1M}(\Omega) + (-1)\lambda\mathcal{D}^1_{-1M}(\Omega)] && \mu \text{ odd}\end{aligned}$$

is proportional to either $\psi_{01M}(\Omega)$ or $\psi_{11M}(\Omega)$ of (1) with $\varepsilon=(-1)^\lambda$ and $\varepsilon'=(-1)^\mu$.

The $\bar{\Gamma}$ operators are observed to leave invariant the subspace of states in $\mathcal{L}^2(\mathrm{SO}(3))$, that belong to a rotor irrep with $\varepsilon=(-1)^\lambda$, $\varepsilon'=(-1)^\mu$. Therefore we may restrict consideration of the matrix elements of $\bar{\Gamma}$ to rotor states of these values of ε and ε'.

The SO(3)-reduced matrix elements of $\bar{\Gamma}$ in the rotor basis (1) are easily determined. Defining reduced matrix elements by means of the Wigner-Eckart theorem

$$\langle K'L'M'|\Gamma_\nu(\mathcal{Q})|KLM\rangle = (2L'+1)^{-1/2}(L2M\nu|L'M')\langle K'L'||\Gamma(\mathcal{Q})||KL\rangle$$

we obtain

$$\begin{aligned}\langle KL'||\Gamma(Q^0)||KL\rangle &= \sqrt{2L+1}(L2K0|L'K)\{\Lambda + \delta_{K1}\sigma_{LL'} + \tfrac{1}{2}L(L+1) - \tfrac{1}{2}L'(L'+1)\}\\ \langle K+2L||\Gamma(Q^0)||KL\rangle &= \sqrt{(2L+1)(1+\delta_{K0})}(L2K2|L'K+2)\sqrt{\tfrac{3}{2}}(\mu-K)\\ \langle K-2L'||\Gamma(Q^0)||KL\rangle &= \sqrt{(2L+1)(1+\delta_{K2})}(L2K-2|L'K-2)\sqrt{\tfrac{3}{2}}(\mu+K)\end{aligned} \tag{10}$$

where $\Lambda = 2\lambda + \mu + 3$ and $\sigma_{LL'}$ is defined by

$$\sigma_{L'L} = \tfrac{1}{2}(\mu+1)(-1)^{\lambda+L} \times \begin{cases} -\dfrac{3L(L+1)}{3-L(L+1)} & \text{for } L' = L \\ L+1 & \text{for } L' = L+1 \\ -L & \text{for } L' = L-1 \\ -1 & \text{for } L' = L \pm 2. \end{cases}$$

These equations immediately imply that the invariant subspace of $\bar{\Gamma}$ containing the state of $L = L_{\min}$ has K values restricted to the range $K = \mu, \mu-2, \ldots, 1$ or 0.

Consider, for example, a $\mu = 0$ irrep. Then $K = 0$ and $\bar{\Gamma}(\mathcal{Q})$ has matrix elements

$$\begin{aligned} \langle 0L+2 \| \bar{\Gamma}(\mathcal{Q}) \| 0L \rangle &= 2(\lambda - L)\left(\frac{3(L+1)(L+2)}{2(2L+3)}\right)^{1/2} \\ \langle 0L \| \bar{\Gamma}(\mathcal{Q}) \| 0L+2 \rangle &= 2(\lambda + L + 3)\left(\frac{3(L+1)(L+2)}{2(2L+3)}\right)^{1/2}. \end{aligned} \tag{11}$$

One sees that, if one steps up in units of two from the lowest angular momentum state of $L = 0$ when λ is even and $L = 1$ when λ is odd, the sequence terminates at $L = \lambda$. Thus we identify the su(3) coherent state subspace for a $(\lambda, 0)$ irrep to be the space spanned by the rotor wavefunctions $\psi_{KLM}(\Omega)$ with $K = 0$ and $L = \lambda, \lambda-2, \ldots, 0$ or 1 and with quadrupole matrix elements given by

$$\langle 0L\pm2 \| \Gamma(\mathcal{Q}) \| 0L \rangle = \langle 0L\pm2 \| \bar{\Gamma}(\mathcal{Q}) \| 0L \rangle \qquad \text{for } L \leqslant \lambda.$$

These results are consistent with the known branching rules (Elliott 1958) for the angular momentum states in an su(3) irrep (λ, μ) given in terms of a convenient multiplicity label k by

$$\begin{aligned} L &= k, k+1, \ldots, k+\lambda \qquad & k \neq 0 \\ &= \lambda, \lambda-2, \ldots, 1 \text{ or } 0 \qquad & k = 0 \\ k &= \mu, \mu-2, \ldots, 1 \text{ or } 0. \end{aligned} \tag{12}$$

To proceed further we need to address two related problems. The first is that for rotor irreps for which there is a multiplicity of states of the same L value, we need a systematic procedure for identifying the combinations of states that belong to the irreducible su(3) subspace. The second problem is that, although the representation Γ is equivalent to a Hermitian representation γ, it is not in fact Hermitian. We therefore seek a similarity transformation

$$\gamma(X) = \mathcal{K}^{-1}\Gamma(X)\mathcal{K} \qquad X \in \text{su}(3) \tag{13}$$

to bring it to its Hermitian form. We shall show that, with minor adaptation, the K-matrix theory (Rowe 1984, Rowe *et al* 1988, Le Blanc and Rowe 1988) solves both problems.

First recall that a representation γ of su(3) is Hermitian if

$$\gamma^{\dagger}(\mathcal{Q}_{\nu}) = (-1)^{\nu}\gamma(\mathcal{Q}_{-\nu}) \qquad \gamma^{\dagger}(L_k) = (-1)^k\gamma(L_{-k}).$$

We therefore define the Hermitian adjoint of a representation Γ by

$$\Gamma^{\#}(\mathcal{Q}_{\nu}) = (-1)^{\nu}\Gamma^{\dagger}(\mathcal{Q}_{-\nu}) \qquad \Gamma^{\#}(L_k) = (-1)^k\Gamma^{\dagger}(L_{-k}).$$

Thus the condition that a representation γ is Hermitian is that

$$\gamma^{\#}(X) = \gamma(X) \qquad \forall X \in \text{su}(3).$$

It follows that, if γ is the Hermitian representation related to the non-Hermitian representation Γ by (13), then by taking Hermitian adjoints we obtain

$$\mathscr{K}^\dagger\Gamma^{\#}(X)=\gamma^{\#}(X)\mathscr{K}^\dagger=\gamma(X)\mathscr{K}^\dagger \qquad \forall X\in \mathrm{su}(3).$$

Multiplying both sides by $\mathscr{K}$, one concludes that Γ and $\Gamma^{\#}$ are equivalent and that the positive Hermitian operator $S=\mathscr{K}\mathscr{K}^\dagger$ is the intertwining operator for which

$$S\Gamma^{\#}(X)=\Gamma(X)S \qquad \forall X\in \mathrm{su}(3). \tag{14}$$

A similar equation exists for the $\bar\Gamma$ and $\bar\Gamma^{\#}$ representations

$$\bar S\bar\Gamma^{\#}(X)=\bar\Gamma(X)\bar S \qquad \forall X\in \mathrm{su}(3). \tag{15}$$

We shall show, by construction, that $\bar S$ is uniquely determined by S. Note that all matrix elements of $\bar\Gamma(\mathscr{Q})$ in the chosen basis are real and that the matrix elements of $\bar\Gamma^{\#}(X)$ are simply related to those of $\bar\Gamma(X)$ by

$$\langle K'L'\|\bar\Gamma^{\#}(\mathscr{Q})\|KL\rangle=(-1)^{L'-L}\langle KL\|\bar\Gamma(\mathscr{Q})\|K'L'\rangle. \tag{16}$$

The transformation $\bar S$ leaves the SO(3)-tensorial properties of states unchanged; i.e.

$$\bar S|KLM\rangle=\sum_{K'}|K'LM\rangle\bar S^L_{K'K}.$$

Thus $\bar S$ is block diagonal in L and the block $\bar S^{L_{\min}}$, for the lowest multiplicity-free value of $L=L_{\min}=0$ or 1, is one dimensional We therefore set $\bar S^{L_{\min}}=1$ and use (15) to define recursion relations for $\bar S^L$ for $L>1$. For example, the operators with $L'=L+1$ or $L+2$ satisfy the equations

$$\langle K'L'\|\bar S^{L'}\bar\Gamma^{\#}(\mathscr{Q})\|KL\rangle=\langle K'L'\|\bar\Gamma(\mathscr{Q})\bar S^{L}\|KL\rangle. \tag{17}$$

If the number of su(3) states with angular momentum L' does not exceed the number with angular momentum L, these equations are sufficient to determine $\bar S^{L'}$ from $\bar S^L$. If there is an extra state of angular momentum L', one then needs the single additional equation

$$\begin{aligned}\sum_{K''}&\langle K'L'\|\bar S^{L'}\bar\Gamma^{\#}(\mathscr{Q})\|K''L'\rangle\langle K''L'\|\bar\Gamma^{\#}(\mathscr{Q})\|K_{\max}L\rangle\\ &=\sum_{K''}\langle K'L'\|\bar\Gamma(\mathscr{Q})\|K''L'\rangle\langle K''L'\|\bar\Gamma(\mathscr{Q})\bar S^L\|K_{\max}L\rangle.\end{aligned} \tag{18}$$

We now claim that the whole problem of constructing the matrices of an arbitrary Hermitian su(3) irrep is reduced to solving these very simple recursion relations for $\bar S^L$ for the values of L that occur in the su(3)$\downarrow$so(3) branching rules of (12). First observe that $\bar S$ maps the space of rotor wavefunction onto its irreducible su(3) (coherent state) subspace. This observation is proved by noting that, while $\bar\Gamma^{\#}(\mathscr{Q})$ generates the whole rotor model space by repeated action on the $|L_{\min}M\rangle$ states, all states of the form

$$P(\Gamma(\mathscr{Q}))|L_{\min}M\rangle=\bar SP(\bar\Gamma^{\#}(\mathscr{Q}))|L_{\min}M\rangle$$

where P is any polynomial, belong to the irreducible su(3) subspace. Furthermore we know that Γ is the restriction of $\bar\Gamma$ and $\Gamma^{\#}$ is the projection of $\bar\Gamma^{\#}$ to the coherent state subspace. We therefore obtain the very strong result that

$$\bar S=S\Pi$$

where Π is the orthogonal projection operator that projects rotor wavefunctions to their coherent state components. The required intertwining operator S of (14) is therefore the positive part of $\bar{S}$. Since S is positive definite, hence invertible, we may define $\mathscr{K}$ to be its positive Hermitian square root and finally obtain the irreducible Hermitian su(3) representation $\gamma(X)$ from (13).

We illustrate with two examples. For a $(\lambda, \mu=0)$ irrep, the recurssion relation (17) for $\bar{S}^L$ together with (16) gives

$$\bar{S}^{L+2}\langle 0L||\bar{\Gamma}(\mathscr{Q})||0L+2\rangle = \langle 0L+2||\bar{\Gamma}(\mathscr{Q})||0L\rangle \bar{S}^L.$$

Hence from equation (11), we immediately obtain

$$\frac{\bar{S}^{L+2}}{\bar{S}^L} = \frac{\lambda - L}{\lambda + L + 3} \qquad \text{for} \qquad L \leqslant \lambda$$

and $\bar{S}^L = 0$ for $L > \lambda$. Therefore

$$\frac{\mathscr{K}^{L+2}}{\mathscr{K}^L} = \left(\frac{\lambda - L}{\lambda + L + 3}\right)^{1/2} \qquad \text{for} \qquad L \leqslant \lambda$$

and the Hermitian matrix elements, obtained from (11), are given by

$$\begin{aligned}
&\langle 0L+2||\gamma(\mathscr{Q})||0L\rangle \\
&\quad = \langle 0L||\gamma(\mathscr{Q})||0L+2\rangle = \langle 0L||\Gamma(\mathscr{Q})||0L+2\rangle \mathscr{K}^{L+2}/\mathscr{K}^L \\
&\quad = 2[(\lambda - L)(\lambda + L + 3)]^{1/2}\left(\frac{3(L+1)(L+2)}{2(2L+3)}\right)^{1/2}.
\end{aligned}$$

For the $(\lambda, \mu) = (2, 2)$ irrep we have the angular momentum states $L = 0, 2, 2, 3, 4$ so that there is now a multiplicity of $L = 2$ states. Setting $\bar{S}^0 = 1$, we obtain from (17) and (18) the four equations for $\bar{S}^2$:

$$\sum_{K'} \bar{S}^2_{KK'}\langle 0||\bar{\Gamma}(\mathscr{Q})||K'2\rangle = \langle K2||\bar{\Gamma}(\mathscr{Q})||0\rangle$$

$$\sum_{K'K''} \bar{S}^2_{KK'}\langle 0||\bar{\Gamma}(\mathscr{Q})||K''2\rangle\langle K''2||\bar{\Gamma}(\mathscr{Q})||K'2\rangle = \sum_{K''}\langle K2||\bar{\Gamma}(\mathscr{Q})||K''2\rangle\langle K''2||\bar{\Gamma}(\mathscr{Q})||0\rangle$$

with $K = 0$ and 2. These equations are easily solved to give

$$\bar{S}^2_{00} = \tfrac{13}{28} \qquad \bar{S}^2_{02} = \bar{S}^2_{20} = \sqrt{3}/28 \qquad \bar{S}^2_{22} = \tfrac{11}{28}.$$

For $\bar{S}^3$ we have the single equation

$$-\bar{S}^3\langle 22||\bar{\Gamma}(\mathscr{Q})||23\rangle = \langle 23||\bar{\Gamma}(\mathscr{Q})||02\rangle \bar{S}^2_{02} + \langle 23||\bar{\Gamma}(\mathscr{Q})||22\rangle \bar{S}^2_{22}$$

with solution $S^3 = \frac{5}{28}$ and, for S^4, we have the four equations

$$\sum_{K'} \bar{S}^4_{KK'}\langle 02||\bar{\Gamma}(\mathscr{Q})||K'4\rangle = \sum_{K'}\langle K4||\bar{\Gamma}(\mathscr{Q})||K'2\rangle \bar{S}^2_{K'0}$$

$$\sum_{K'} \bar{S}^4_{KK'}\langle 22||\bar{\Gamma}(\mathscr{Q})||K'4\rangle = \sum_{K'}\langle K4||\bar{\Gamma}(\mathscr{Q})||K'2\rangle \bar{S}^2_{K'2}$$

with solution

$$\bar{S}^4_{00} = \tfrac{1}{21} \qquad \bar{S}^4_{02} = \bar{S}^4_{20} = \sqrt{5}/42 \qquad \bar{S}^4_{22} = \tfrac{5}{84}.$$

Note that the su(3) subspace for the (2, 2) irrep has a single $L = 4$ state which has to be projected from among the span of the three $K = 0, 2, 4$ states of the corresponding rotor representation. The $K=4$ state is immediately projected out since the su(3) subspace has only states with $K \leqslant \mu$. From the solution for $\bar{S}^4$, we determine that

$$\bar{S}|04M\rangle = \sum_{K=0}^{2} |K4M\rangle \bar{S}^4_{K0} = \tfrac{1}{21}[|04M\rangle + (\sqrt{5}/2)|24M\rangle]$$

$$\bar{S}|24M\rangle = \sum_{K=0}^{2} |K4M\rangle \bar{S}^4_{K2} = \sqrt{5}/42[|04M\rangle + (\sqrt{5}/2)|24M\rangle].$$

Thus one finds that $\bar{S}$ projects both rotor states onto the single $L=4$ su(3) state

$$|4M\rangle = |04M\rangle + (\sqrt{5}/2)|24M\rangle$$

and that

$$S|4M\rangle = \tfrac{3}{28}|4M\rangle.$$

From the S^L matrices one readily determines the $\mathcal{K}^L = \sqrt{S^L}$ matrices. Finally one computes the desired matrix elements

$$\langle k'L'||\gamma(\mathcal{Q})||kL\rangle = \langle k'L'||(\mathcal{K}^{L'})^{-1}\Gamma(\mathcal{Q})\mathcal{K}^L||kL\rangle$$

in this basis.

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